Lecture 34

- **Note for Exam**
  - Orthogonality

    (1) Turn a basis $\{\vec{v}_1, \dots, \vec{v}_k\}$ for a subspace W into an orthogonal basis.

    (2) Find $\perp$ proj of $\vec{v}$ onto W

    $$\text{proj}_W \vec{v} = \text{proj}_{\vec{v}_1}(v) + \dots + \text{proj}_{\vec{v}_k}(v)$$

    ↳ only works if $\vec{v}_1, \dots, \vec{v}_k$ is an $\perp$ basis.

- $A \in M_{n \times n}$
- $\lambda \in \mathbb{R}$ is an e-val of $A \leftrightarrow \lambda$ is a root of the characteristic polynomial of A.
- $\vec{v} \in \mathbb{R}$ is an e-vec with e-val $\lambda \leftrightarrow \vec{v} \in \ker(A - \lambda I)$

Def: The λ-eigenspace of an e-val $\lambda \in \mathbb{R}$ for a matrix A is the set of all e-vecs of A with e-val λ.

E.g.

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -7 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -7 \end{bmatrix}$$

- Find e-vals, e-vecs
- For both of A and B we have $\det(A - \lambda I) = 0$

$$\det(A - \lambda I) = \begin{vmatrix} 2-\lambda & 0 & 0 \\ 0 & 2-\lambda & 0 \\ 0 & 0 & -7-\lambda \end{vmatrix} = (2-\lambda)^2(-7-\lambda)$$

Same characteristic polynomial for A and B: $(2-\lambda)^2(-7-\lambda) = 0$

→ so they have the same e-vals : $2, 2, -7$

↳ i.e. 2 is an eval with algebraic multiplicity 2

↳ i.e. -7 is an eval with algebraic multiplicity 1

What about e-vecs?

For $\lambda = -7$ so $[B - (-7)I \mid \vec{0}] \to [B + 7I \mid 0] \to \left[\begin{array}{ccc|c} 9 & 1 & 0 & 0 \\ 0 & 9 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$$\to \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

So $\vec{v} = x_3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ i.e. the (-7) eigenspace of B is the span$(\vec{e}_3)$

<u>In fact</u>: The (-7)-eigenspace of $A = \text{span}(\vec{e}_3)$

- In particular the <u>geometric</u> <u>multiplicity</u> of (-7) is $\dim[\text{span}(\vec{e}_3)] = 1$

- <u>OTOH</u>: $\lambda = 2$: $[A - 2\lambda \mid 0] \to \left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -9 & 0 \end{array}\right] \to \vec{v} = x_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \vec{x}_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

- So 2-eigenspace of A is span $(\vec{e}_1, \vec{e}_2) \to 2$ has geometric multiplicity 2.

But for B, we get

$$[B - 2I \mid \vec{0}] = \left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -7 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\to \vec{v} = x_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \to$ 2-espace of $B = \text{span}(\vec{e}_1)$

$\to$ geometric multiplicity of 1

<u>Moral</u>: Typically the algebraic and geometric multiplicities of an e-val are equal, but geometric multiplicity can be less.

$\boxed{\text{Important for Final}}$

easier e.g. than $B : \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

1 eval with algebraic multiplicity = 2, but geometric multiplicity 1.

- **Thm**: If $\lambda \in \mathbb{R}$ is an e-val of $A \in M_{n \times n}$, then geometric multiplicity of $\lambda \leq$ algebraic multiplicity.

Pf: Consider $T: \mathbb{R}^n \to \mathbb{R}^m$ given by $T(\vec{x}) = A\vec{x}$.

- Suppose $\lambda \in \mathbb{R}$ is an e-val with geometric mult $= k$.
  - Then $\exists$ basis $\{\vec{v}_1, \dots, \vec{v}_k\}$ for the λ-eigenspace of A.
  - I choose vectors $\vec{v}_{k+1}, \dots, \vec{v}_n$ that extend this basis to a basis $\beta = \{\vec{v}_1, \dots, \vec{v}_n\}$ for $\mathbb{R}^n$

- Then $[T]_\beta = \begin{bmatrix} [T(\vec{v}_1)]_\beta & \cdots & [T(\vec{v}_n)]_\beta \end{bmatrix}$

$$= \begin{bmatrix} [\lambda v_1]_\beta & \cdots & [\lambda v_k]_\beta & * & \cdots & * \end{bmatrix}$$

$$= \begin{bmatrix} \underbrace{\begin{matrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ \vdots & 0 & \lambda \\ 0 & \vdots & \vdots \\ & 0 & 0 \end{matrix}}_{k \sim \text{cols}} & \cdots & \underbrace{*}_{\text{no-clue}} \end{bmatrix}$$

- So characteristic polynomial of $[T]_\beta = \det([T]_\beta - tI)$

$$= \left[\begin{array}{ccc|c} \lambda - t & 0 & 0 & \\ 0 & \lambda - t & 0 & * \\ 0 & 0 & \lambda - t & \\ \vdots & \vdots & \vdots & \\ \hline 0 & 0 & 0 & * \end{array}\right]$$

· got lost from here

but $\det([T]_\beta - \lambda I) = (\lambda - t)^k \cdot$ some polynomial

→ so alg multiplicity λ is $\geq k$ (geo mult of λ)

$\boxtimes$

<u>Last time</u>: $A \in M_{n \times n}$ is diagonalizable if the roots of the characteristic polynomial $[\det(A - \lambda I) = 0]$ are all real and distinct.

Reason: e-vecs for different e-vals are L.I.

<u>Thm</u>: A matrix $A \in M_{n \times n}$ is diagonalizable i.f.f. all roots of $\det(A - \lambda I)$ are real and the geometric multiplicity of each is equal to its alg multiplicity.

<u>Why?</u>

· $\det(A - \lambda I)$ = polynomial of degree n → n possibly complex roots counted with algebraic multiplicity.

So if all roots are real and the geo mult of each = its alg mult → n L.I. e-vecs.